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A note on the steady two-dimensional flow of a stratified fluid over an obstacle

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The steady two-dimensional flow of an inviscid incompressible fluid of variable density is considered in a long channel, bounded above by a rigid horizontal plane and below by an obstacle. For certain variations with height of the speed and density in the incident stream, the governing equation is the reduced wave equation. Drazin & Moore (1967) have recently used this fact to develop a wave-guide analogy. In this note the wave-guide analogy is further developed and several uniqueness theorems obtained. When the obstacle satisfies a certain convexity condition it is shown that the upstream conditions and the obstacle uniquely determine the flow; that is, there is no critical internal Froude number or obstacle height for which the problem fails to be well posed.

1. Introduction

The flow past an obstacle of a horizontal stream of fluid of variable density and speed and subject to a gravitational force is of considerable importance in geophysics. Long (1953, 1954, 1955) has made the important discovery that for the steady, two-dimensional flow of an inviscid, incompressible fluid the governing equation becomes the reduced wave equation (i.e. Helmholtz's equation) for certain variations with height of the speed and density in the incident stream. As Drazin & Moore (1967) have recently pointed out, this makes available the powerful techniques of diffraction theory.

The flow is in a long channel, bounded above by a rigid horizontal plane and below by an obstacle; far upstream it will be assumed that the flow is unperturbed. This 'lee-wave' condition has not been universally accepted; Long (1955) himself has suggested that for sufficiently small internal Froude numbers (cf. (2.5), i.e. intense stratification), or for sufficiently high obstacles, the disturbance might extend far upstream and so 'block' the flow. However, Long's argument assumes that the flow can be continued analytically into the interior of the obstacle, whereas Yih (1960) and others have shown that flows can be constructed by placing singularities within the obstacle. Trustrum (1964) considered the development of an unsteady flow using an 'Oseen' type linearization, and her results support the possibility of blocking. On the other hand, Drazin & Moore (1967) have used a wave-guide analogy to suggest that the 'lee-wave' condition makes the problem a well-posed one, and that 'blocking' is associated with an energy restriction (Sheppard 1956).

In this note the wave-guide analogy is further developed and some of the conjectures of Drazin & Moore (1967) are established. In §2 the problem is formulated and the equation of motion and the boundary conditions are stated. In §3 it is shown that the stratified flow problem may be constructed in a unique way from a set of associated wave-guide problems, and that the downstream lee-waves are uniquely determined by the obstacle and the upstream conditions. In §4 two uniqueness theorems are presented. First, a result of Long's (1953) is improved to show that for sufficiently large Froude numbers (i.e. sufficiently small k, see (2.5)) the flow is uniquely determined. Secondly, it is shown that if the obstacle satisfies a certain convexity condition (see (4.7)) then the flow is uniquely determined by the upstream conditions and the obstacle; this result holds for all Froude numbers and is independent of the height of the obstacle. The convexity condition (4.7) includes all obstacles so far treated (e.g. Yih 1965). The flow is assumed to be smooth so that the possibility of a stagnant region occurring is explicitly excluded (Kao 1965); also the fact that the flow is assumed steady and the 'lee-wave' condition has been used prevents a direct conflict with the work of Trustrum (1964). Our concern has been to establish when the 'lee-wave' condition, together with Long's equation, presents a well-posed problem in the mathematical sense. No attempt is made to establish existence theorems, although the validity of the wave-guide analogy supports the conjecture that solutions will exist for all Froude numbers and obstacle heights.

2. Equation of motion

We consider the two-dimensional steady flow of inviscid incompressible fluid of variable density ρ^* (with velocity $\mathbf{v}^*$ and pressure p^* at the point $\mathbf{r}^*$). The stream runs in a long channel whose upper boundary is the rigid horizontal plane $y^* = d$ and whose lower boundary is the rigid obstacle $y^* = h^*(x^*)$, where $h^*(x^*) = 0$ for sufficiently large $|x^*|$. Far upstream (i.e. as $x^* \to -\infty$) the flow is horizontal (and in the positive x^*-direction) with a prescribed speed $U^*_\infty(y^*)$ and density $\rho^*_\infty(y^*)$.

If U, ρ_0 are characteristic scales for the upstream speed and density, we introduce the dimensionless variables

$$\mathbf{r} = \frac{\pi \mathbf{r}^*}{d}, \quad \mathbf{v} = \left(\frac{\rho^*}{\rho_0}\right)^{\frac{1}{2}} \frac{\mathbf{v}^*}{U}, \quad \rho = \frac{\rho^*}{\rho_0}, \quad p = \frac{p^*}{\rho_0 U^2}, \tag{2.1}$$

where in the second equation, we have incorporated Yih's transformation (cf. Yih 1965, p. 5). Since the flow is two-dimensional, a stream function $\psi(x,y)$ may be defined so that

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x, \tag{2.2}$$

where u and v are the x- and y-components respectively of $\mathbf{v}$. The density and the head H^*, where $H^* = p^* + \frac{1}{2}\rho^*|\mathbf{v}^*|^2 + g\rho^* y^*$, are functions of ψ only and are to be determined from the prescribed conditions upstream. Following Long (1955) we choose these conditions so that

$$\rho^*_\infty (U^*_\infty)^2 = \rho_0 U^2, \quad d\rho^*_\infty/dy^* = -\beta\rho_0, \tag{2.3}$$

where β is a constant. The equation governing balance of momentum is then the linear equation

$$\nabla^2\psi + k^2(\psi - y) = 0, \tag{2.4}$$

where

$$k^2 = g\beta d^2/U^2\pi^2 \tag{2.5}$$

is the reciprocal of an internal Froude number (cf. Yih 1965, p. 83). For the incident stream to be stable, β and hence k^2 are positive. If

$$\phi = y - \psi, \tag{2.6}$$

then ϕ satisfies the reduced wave equation

$$\nabla^2\phi + k^2\phi = 0. \tag{2.7}$$

The top $(y = \pi)$ and bottom $\left(y = h(x), \text{ where } h^*(x^*) = \frac{d}{\pi}h(x)\right)$ of the channel are the streamlines $\psi = \pi$ and $\psi = 0$ respectively. Far upstream the conditions (2.3) must be satisfied, so that $\psi \to y$ as $x \to -\infty$; far downstream ψ is to be bounded. These lead to the following boundary conditions for ϕ:

$$\phi = 0 \quad \text{when} \quad y = \pi, \qquad \phi = h(x) \quad \text{when} \quad y = h(x), \tag{2.8}$$

$$\phi \to 0 \quad \text{as} \quad x \to -\infty, \qquad \phi \text{ is bounded} \quad \text{as} \quad x \to +\infty. \tag{2.9}$$

3. Associated wave-guide problems

We shall assume that the solutions of (2.7) are complex-valued functions which are twice continuously differentiable within the channel and whose first derivatives are continuous up to the channel boundaries. Let $h(x) = 0$ for $|x| > L_0$; then in these regions the solutions may be obtained from Fourier analysis:

$$\phi = \sum_1^\infty \phi_n^\pm(x)\sin ny \quad \text{for} \quad |x| > L_0, \tag{3.1}$$

where

$$\phi_n^\pm(x) = \frac{2}{\pi}\int_0^\pi \phi(x, y')\sin ny'\,dy', \tag{3.2}$$

and ϕ_n^+, ϕ_n^- refer to $x > L_0$ and $x < -L_0$ respectively. The Fourier series for the derivatives of ϕ may be obtained by differentiating (3.1) term-by-term; all the series involved are uniformly convergent in y, for each fixed x. If (2.7) is multiplied by $\sin ny$ and then integrated with respect to y from 0 to π, the following equations are obtained:

$$\frac{d^2\phi_n^\pm}{dx^2} + (k^2 - n^2)\phi_n^\pm = 0 \quad (n = 1, 2, 3, \ldots). \tag{3.3}$$

Then if $K < k < K+1$ for some non-negative integer K, (3.3) has the solutions

$$\phi_n^\pm = A_n^\pm \cos\{(k^2-n^2)^{\frac{1}{2}}x\} + B_n^\pm \sin\{(k^2-n^2)^{\frac{1}{2}}x\} \quad (1 \leqslant n \leqslant K), \tag{3.4}$$

and

$$\phi_n^\pm = A_n^\pm \exp\{-(n^2-k^2)^{\frac{1}{2}}|x|\} + B_n^\pm \exp\{(n^2-k^2)^{\frac{1}{2}}|x|\} \quad (n \geqslant K+1), \tag{3.5}$$

where $A_n^\pm$, $B_n^\pm$ are arbitrary constants. The sinusoidal solutions (3.4) are said to be subcritical and the exponential solutions supercritical (cf. Benjamin 1966). The cases where k is an integer correspond to wave-guide resonance between the walls $y = 0, \pi$; the flow is said to be critical and no steady flow can be anticipated. These cases are therefore excluded.

Application of the upstream condition (2.9) shows that $A_n^- = B_n^- = 0$ for $1 \leqslant n \leqslant K$, and $B_n^- = 0$ for $n \geqslant K+1$; the downstream condition shows that $B_n^+ = 0$ for $n \geqslant K+1$.

Following Drazin & Moore (1967) we propose to treat the stratified flow problem of § 2 by developing a wave-guide analogy, although the treatment given here differs in some respects from theirs. We define

$$W_n, V_n = \exp\{\pm i(k^2-n^2)^{\frac{1}{2}}|x|\}\sin ny \quad (1 \leqslant n \leqslant K), \tag{3.6}$$

$$E_n = \exp\{-(n^2-k^2)^{\frac{1}{2}}|x|\}\sin ny \quad (n \geqslant K+1), \tag{3.7}$$

where W_n, V_n correspond to outgoing and incoming waves respectively. Next we introduce the following wave-guide emission and scattering problems.

(E): $\phi^{(E)}$ is a complex-valued solution of (2.7) which satisfies the boundary conditions

$$\phi^{(E)} = 0 \quad \text{when} \quad y = \pi, \qquad \phi^{(E)} = h(x) \quad \text{when} \quad y = h(x), \tag{3.8a}$$

$$\phi^{(E)} = \sum_{n=1}^{K} B_n^{\pm} W_n + \sum_{n=K+1}^{\infty} B_n^{\pm} E_n \quad \text{for} \quad |x| > L_0; \tag{3.8b}$$

(S_m), $(1 \leqslant m \leqslant K)$: $\phi^{(m)}$ is a complex-valued solution of (2.7) which satisfies the boundary conditions

$$\phi^{(m)} = 0 \quad \text{when} \quad y = \pi \quad \text{and when} \quad y = h(x), \tag{3.9a}$$

$$\left.\begin{aligned} \phi^{(m)} &= \sum_{n=1}^{K} A_{mn} V_n + \sum_{n=1}^{K} C_{mn}^{+} W_n + \sum_{n=K+1}^{\infty} C_{mn}^{+} E_n \quad \text{for} \quad x > L_0, \\ \phi^{(m)} &= W_m + \sum_{n=K+1}^{\infty} C_{mn}^{-} E_n \quad \text{for} \quad x < -L_0, \end{aligned}\right\} \tag{3.9b}$$

where A_{mn}, $B_n^{\pm}$, $C_{mn}^{\pm}$ are constants which are to be determined. The condition (3.8b) is the appropriate form of the Sommerfeld radiation condition in this context, and ensures that all waves are outgoing at infinity. The condition (3.9b) is atypical for a scattering problem; however, it is the most convenient for our purposes, and we shall show later that it is equivalent to the more usual scattering problem in which the incident wave is prescribed and it is the outgoing waves which are to be determined. If we put

$$\phi^{(F)} = \phi^{(E)} - \sum_{m=1}^{K} B_m^{-} \phi^{(m)}, \tag{3.10}$$

then $\phi^{(F)}$ has no outgoing (or incoming) waves in the region $x < -L_0$, and since the governing equation (2.7) and the boundary conditions (3.9) are linear, is clearly a solution to the stratified flow problem of § 2 (cf. Drazin & Moore 1967). Clearly (3.10) is the only linear combination of $\phi^{(E)}$ and $\phi^{(m)}$ which has no waves in $x < -L_0$ (since the functions W_n are linearly independent), and so $\phi^{(F)}$ will be uniquely determined by (2.8) and (2.9) when $\phi^{(E)}$ and $\phi^{(m)}$ are uniquely determined by (3.8) and (3.9) respectively.

If ϕ is a complex-valued solution of (2.7) in a region S, whose boundary C is piecewise smooth, then an application of the Gauss divergence theorem shows that

$$\oint_C \left(\phi \frac{\partial \bar{\phi}}{\partial \nu} - \bar{\phi} \frac{\partial \phi}{\partial \nu}\right) dl = 0, \tag{3.11}$$

where $\partial/\partial\nu$ denotes differentiation along the outward normal to S. Let S be that part of the channel for which $|x| < L$, where $L > L_0$ (see figure 1), and let ϕ satisfy the following boundary conditions:

$$\phi = 0 \quad \text{when} \quad y = \pi \quad \text{and when} \quad y = h(x), \tag{3.12a}$$

$$\left.\begin{aligned} \phi &= \sum_{n=1}^{K} A_n V_n + \sum_{n=1}^{K} D_n^+ W_n + \sum_{n=K+1}^{\infty} D_n^+ E_n \quad \text{for} \quad x > L_0, \\ \phi &= \sum_{n=1}^{K} D_n^- W_n + \sum_{n=K+1}^{\infty} D_n^- E_n \quad \text{for} \quad x < -L_0. \end{aligned}\right\} \tag{3.12b}$$

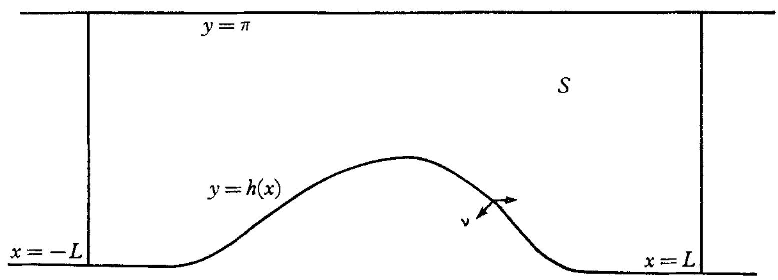

FIGURE 1. The region of integration S, for a typical obstacle which satisfies the convexity condition.

Then the integrals over C reduce to integrals over the strips $|x| = L$, and these may be evaluated by applying Parseval's formula for Fourier series (e.g. Zygmund 1959, pp. 13, 37); for reasons of economy of space only one such calculation is included here:

for $x = L$
$$\int_0^\pi \phi \frac{\partial\bar{\phi}}{\partial x}\,dy = \frac{\pi}{2}\sum_{n=1}^{\infty} \phi_n^+ \frac{d\bar{\phi}_n^+}{dx}$$

$$= \frac{i\pi}{2}\sum_{n=1}^{K} (k^2-n^2)^{\frac{1}{2}}[|A_n|^2 - |D_n^+|^2 + \bar{A}_n D_n^+ \exp\{2i(k^2-n^2)^{\frac{1}{2}}L\} - A_n \bar{D}_n^+ \exp\{-2i(k^2-n^2)^{\frac{1}{2}}L\}]$$
$$-\frac{\pi}{2}\sum_{n=K+1}^{\infty} (n^2-k^2)^{\frac{1}{2}}|D_n^+|^2 \exp\{-2(n^2-k^2)^{\frac{1}{2}}L\}. \tag{3.13}$$

Thus it follows from (3.11) and (3.12), using expressions such as (3.13) that

$$\sum_{n=1}^{K} (k^2-n^2)^{\frac{1}{2}}|A_n|^2 = \sum_{n=1}^{K} (k^2-n^2)^{\frac{1}{2}}(|D_n^+|^2 + |D_n^-|^2). \tag{3.14}$$

This equation expresses conservation of energy in the wave-guide.

To establish uniqueness for the emission problem (E) it must be shown that if ϕ is a complex-valued solution to (2.7) which satisfies (3.12a) and (3.12b) with $A_n = 0$, then ϕ is identically zero. But when $A_n = 0$, it follows from (3.14) that $D_n^+ = D_n^- = 0$. Thus the coefficients $B_n^\pm$ for $1 \leqslant n \leqslant K$ of (3.8b) are uniquely determined by the bottom topography. We defer until the next section the question of when $\phi^{(E)}$ (and hence the remaining $B_n^\pm$) are uniquely determined.

Next we turn our attention to the scattering problems (S_m). If $\{F_m\}$ ($1 \leqslant m \leqslant K$) are a set of arbitrary constants, put

$$\phi = \sum_{m=1}^{K} F_m \phi^{(m)}.$$

Then

$$A_n = \sum_{m=1}^{K} A_{mn} F_m, \quad D_n^+ = \sum_{m=1}^{K} C_{mn}^+ F_m, \quad D_n^- = F_n \quad \text{for} \quad 1 \leqslant n \leqslant K,$$

and substitution into (3.14) shows that

$$\sum_{n=1}^{K} (k^2 - n^2)^{\frac{1}{2}} \left| \sum_{m=1}^{K} A_{mn} F_m \right|^2 \geqslant \sum_{n=1}^{K} (k^2 - n^2)^{\frac{1}{2}} |F_n|^2. \tag{3.15}$$

Thus the set of coefficients

$$\left\{ \sum_{m=1}^{K} A_{mn} F_m \right\} \quad (1 \leqslant n \leqslant K)$$

cannot all vanish unless all the members of the set $\{F_n\}$ are zero. It follows that

$$\det [A_{mn}] \neq 0 \tag{3.16}$$

and so the inverse of the matrix $[A_{mn}]$ exists; we let A_{mn}^{-1} denote the components of this inverse. One consequence of (3.16) is that there is a unique set of coefficients $\{F_m\}$ such that the set

$$\left\{ \sum_{m=1}^{K} A_{mn} F_m \right\} = \{\delta_{nr}\}$$

for some r, $1 \leqslant r \leqslant K$; this establishes the equivalence of the problems S_m to the scattering problem when an incident wave V_r is prescribed in $x > L_0$.

Now suppose that $\tilde{\phi}^{(m)}$ is also a solution of S_m with coefficients $\tilde{A}_{mn}$, $\tilde{C}_{mn}^{\pm}$, and that ϕ is the difference between two solutions of the scattering problem when the incident wave V_r is prescribed in $x > L_0$. Thus

$$\phi = \sum_{m=1}^{K} A_{rm}^{-1} \phi^{(m)} - \sum_{m=1}^{K} \tilde{A}_{rm}^{-1} \tilde{\phi}^{(m)} \quad (1 \leqslant r \leqslant K),$$

so that

$$A_n = 0, \quad D_n^+ = \sum_{m=1}^{K} A_{rm}^{-1} C_{mn}^+ - \sum_{m=1}^{K} \tilde{A}_{rm}^{-1} \tilde{C}_{mn}^+, \quad D_n^- = A_{rn}^{-1} - \tilde{A}_{rn}^{-1} \quad \text{for} \quad 1 \leqslant n \leqslant K.$$

But when $A_n = 0$, it follows from (3.14) that $D_n^+ = D_n^- = 0$, and so

$$A_{rn} = \tilde{A}_{rn}, \quad \sum_{m=1}^{K} (C_{mn}^+ - \tilde{C}_{mn}^+) A_{rm}^{-1} = 0,$$

or

$$C_{rn}^+ = \tilde{C}_{rn}^+ \quad \text{for} \quad 1 \leqslant r, n \leqslant K$$

since the matrix $[A_{mn}]$ is non-singular. Thus the wave terms of $\phi^{(m)}$ are uniquely determined; it follows from (3.10) that the lee-wave components (i.e. the sinusoidal terms (3.4)) of $\phi^{(F)}$ are uniquely determined by the upstream conditions and the bottom topography.

The analysis so far demonstrates the validity of the wave-guide analogy, thus fulfilling a conjecture of Drazin & Moore (1967). It has also been shown that the solution to (E) and (S_m) (and hence to the stratified flow problem) will be unique if and only if there are no eigensolutions; we call ϕ an eigensolution if it is a solution of the following problem

(U): ϕ is a non-zero complex-valued solution of (2.7) which satisfies the boundary conditions

$$\phi = 0 \quad \text{when} \quad y = \pi \quad \text{and when} \quad y = h(x), \tag{3.17a}$$

$$\phi = \sum_{n=K+1}^{\infty} A_n^{\pm} E_n \quad \text{for} \quad |x| > L_0, \tag{3.17b}$$

for some constants $A_n^{\pm}$.

4. Uniqueness theorems

It is to be expected that, in general, eigensolutions will exist. Thus to establish uniqueness, further restrictions must be placed on one or both of k and $h(x)$. Two theorems are presented here: first, if k is suitably small then there is uniqueness for any bottom topography (provided that $h(x) = 0$ for $|x| > L_0$); secondly, if the obstacle satisfies the convexity condition (4.7) then there is uniqueness for all (non-integral) k.

First we consider the case when $0 \leqslant k < 1$ (i.e. $K = 0$). We commence with the identity (cf. Long 1953)

$$\iint_S |\operatorname{grad}\phi - \phi \operatorname{grad}\eta|^2 dS - \iint |\phi|^2 \{k^2 + \nabla^2\eta + |\operatorname{grad}\eta|^2\} dS = \frac{1}{2}\oint_C^S \left(\phi\frac{\partial\bar{\phi}}{\partial\nu} + \bar{\phi}\frac{\partial\phi}{\partial\nu}\right) dl - \oint_C |\phi|^2 \frac{\partial\eta}{\partial\nu}\, dl, \tag{4.1}$$

where ϕ is a complex-valued solution of (2.7) in a region S whose boundary C is piece-wise smooth, and η is a twice continuously differentiable real function in S whose first derivatives are continuous up to C. Let S be that part of the channel for which $|x| < L$, where $L > L_0$, and let ϕ be a solution of (U). Then the integrals over C reduce to integrals over the strips $|x| = L$, and an analysis similar to that leading to (3.14) shows that

$$\oint_C \left(\phi\frac{\partial\bar{\phi}}{\partial\nu} + \bar{\phi}\frac{\partial\phi}{\partial\nu}\right) dl = -\pi \sum_{n=K+1}^{\infty} (n^2-k^2)^{\frac{1}{2}}(|A_n^+|^2 + |A_n^-|^2)\exp\{-2(n^2-k^2)^{\frac{1}{2}}L\}. \tag{4.2}$$

k is now restricted so that $\quad 0 \leqslant k(\pi - y_m) < \pi(1-\epsilon), \tag{4.3}$

where
$$y_m = \min_{|x|<L_0} (0, h(x)),$$

and where ϵ is a small positive constant chosen so that $k < 1-\epsilon$; note that if $h(x) \geqslant 0$ then $y_m = 0$. If we put

$$\eta = \log\{\sin(k(y-y_m) + \pi\epsilon)\}, \tag{4.4}$$

then η is twice continuously differentiable in S, $\partial\eta/\partial x \equiv 0$ and

$$k^2 + \nabla^2\eta + |\operatorname{grad}\eta|^2 \equiv 0 \quad \text{for all} \quad x, y_m \leqslant y \leqslant \pi. \tag{4.5}$$

The second integral on the left-hand side of (4.1) now vanishes identically (this device is due to Long (1953); see also Frank & von Mises (1961, p. 784). Since the second integral on the right-hand side of (4.1) is also identically zero, it follows from (4.2) that the right-hand side of (4.1) is non-positive, and hence

$$\iint_S |\operatorname{grad}\phi - \phi \operatorname{grad}\eta|^2 dS = 0. \tag{4.6}$$

From this equation it may easily be shown that ϕ is identically zero. Thus when k is small enough to satisfy (4.3), the upstream conditions and the bottom topography uniquely determine the flow in the entire channel. This result† is an improved version of a theorem of Long (1953), who considered solutions of (2.7) in a channel of finite length, with conditions of periodicity at the ends.

Next we turn our attention to the case when k is unrestricted (but not an integer), and $h(x)$ satisfies the following convexity condition. We choose the origin of x so that $h(0)$ is a maximum of $h(x)$, and require that

$$x\,\partial x/\partial \nu \leqslant 0, \tag{4.7}$$

i.e. the outward normal to the obstacle makes an angle of not less than 90° with the unit vector which is parallel to the x-axis, and points upstream for $x < 0$ and downstream for $x > 0$ (see figure 1). We commence with the identity

$$\iint_S \frac{\partial F}{\partial x}\left(k^2|\phi|^2 - |\operatorname{grad}\phi|^2 + 2\left|\frac{\partial \phi}{\partial x}\right|^2\right) + F\left(\frac{\partial \overline{\phi}}{\partial x}(\nabla^2\phi + k^2\phi) + \frac{\partial \phi}{\partial x}(\nabla^2\overline{\phi} + k^2\overline{\phi})\right) dS$$
$$= \oint_C F\left(\frac{\partial x}{\partial \nu}(k^2|\phi|^2 - |\operatorname{grad}\phi|^2) + \frac{\partial \phi}{\partial x}\frac{\partial \overline{\phi}}{\partial \nu} + \frac{\partial \overline{\phi}}{\partial x}\frac{\partial \phi}{\partial \nu}\right) dl, \tag{4.8}$$

where C is the piece-wise smooth boundary of S, ϕ is a twice continuously differentiable function in S whose first derivatives are continuous up to C, and F is a differentiable function of x only which is continuous up to C (cf. similar identities used by Rellich (1943, 1948, p. 329), and Jones (1953)). Let S be that part of the channel for which $|x| < L$, and let ϕ be a solution of (U); if we put $F = x$ and combine (4.8) with (4.1) when $\eta \equiv 0$, then

$$2\iint_S \left|\frac{\partial \phi}{\partial x}\right|^2 dS = \frac{1}{2}\int_{|x|=L}\left(\phi\frac{\partial \overline{\phi}}{\partial |x|} + \overline{\phi}\frac{\partial \phi}{\partial |x|}\right) dy$$
$$+ \int_{|x|=L} |x|\left(k^2|\phi|^2 - |\operatorname{grad}\phi|^2 + 2\left|\frac{\partial \phi}{\partial |x|}\right|^2\right) dy + \int_{y=h(x)} x\frac{\partial x}{\partial \nu}\left|\frac{\partial \phi}{\partial \nu}\right|^2 dl. \tag{4.9}$$

From (4.2) the first integral on the right-hand side of (4.9) is non-positive, and from the convexity condition (4.7) the third integral is also non-positive. The second integral is then evaluated from Parseval's formula, and it is found that

$$\int_{|x|=L}\left(k^2|\phi|^2 - |\operatorname{grad}\phi|^2 + 2\left|\frac{\partial \phi}{\partial |x|}\right|^2\right) dy = 0. \tag{4.10}$$

Indeed this integral contains only the coefficients corresponding to wave terms in ϕ, and these are all absent when ϕ is a solution of (U). Hence

$$\iint_S \left|\frac{\partial \phi}{\partial x}\right|^2 dS = 0 \tag{4.11}$$

† This uniqueness theorem has also been obtained by Drazin & Moore (unpublished) for the case $h(x) \geqslant 0$, using a different method.

and it follows easily that ϕ is identically zero. Thus when the obstacle satisfies the convexity condition (4.7) the flow in the entire channel is uniquely determined by the upstream conditions and the bottom topography, for all (non-integral) values of k; also the uniqueness holds no matter how high the obstacle is, or how long.

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